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# Asymptotic theory for estimating drift parameters in the fractional Vasicek model

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# Asymptotic Theory for Estimating Drift Parameters in the Fractional Vasicek Model*

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**Abstract**

This paper develops the asymptotic theory for estimators of two parameters in the drift function in the fractional Vasicek model when a continuous record of observations is available. The fractional Vasicek model is assumed to be driven by the fractional Brownian motion with a known Hurst parameter greater than or equal to one half. It is shown that the asymptotic theory for the persistent parameter depends critically on its sign, corresponding asymptotically to the stationary case, the explosive case, and the null recurrent case. In all three cases, the least squares method is considered. When the persistent parameter is positive, the estimate method of Hu and Nualart (2010) is also considered. The strong consistency and the asymptotic distribution are obtained in all three cases.

**JEL Classification:** C15, C22, C32.

**Keywords:** Least squares; Fractional Vasicek model; Stationary process; Explosive process; Null recurrent; Strong consistency; Asymptotic distribution

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# 1 Introduction

The Vasicek model of (36) has found a wide range of applications in many fields, including but not limited to economics, finance, biology, physics, chemistry, medicine and environmental studies. An intrinsic property implied by the standard Vasicek model is the short-range dependence of the stochastic component of the model because the autocovariance decays in a geometric rate. This property is at odd with abundant empirical evidence that indicates the long-range dependence or long memory in time series data (see e.g. (22; 10; 17)). As a result, stochastic models with long-range dependence have been used to describe the movement of time series data in hydrology, geophysics, climatology and telecommunication, economics and finance.

In continuous time, the factional Brownian motion (fBm) is an important stochastic process to characterize the long-range dependence (see e.g. (23)). The fBm produces burstiness, self-similarity and stationary increments in the sample path. Excellent surveys on fBm can be found in (5) and (24).

If the Brownian motion in the Vasicek model is replaced with fBm, we get the following fractional Vasicek model (fVm)

$$dX_t = \kappa\left(\mu - X_t\right)dt + \sigma dB_t^H, \tag{1.1}$$

where $\sigma$ is a positive constant, $\mu, \kappa \in \mathbb{R}$, and $B_t^H$ is fBm (which will be defined formally below) with a known Hurst parameter $H \geq 1/2$. $T$he long-range dependence in $X_t$ is generated by $B_t^H$.

In Model (1.1), $\kappa\left(\mu - X_t\right)$ is the drift function and there are two unknown parameters in it, $\mu$ and $\kappa$. Parameter $\kappa$ determines the persistence in $X_t$. Depending on the sign of $\kappa$, the model can capture the stationary, the explosive, and the null recurrent behavior. The fVm was first used to describe the dynamics in volatility by (11). Other applications of fVm can be found in (9; 7; 8; 12; 2) and references therein. Despite many applications of fVm in practice, to the best of our knowledge, estimation and the asymptotic theory in fVm has received little attention in the literature. The main purpose of the present paper is to propose estimators for $\mu$ and $\kappa$ and to develop the asymptotic theory for the estimators.

A very important special case of fVm is the so-called fractional Ornstein-Uhlenbeck (fOU) process given by:

$$dX_t = -\kappa X_t dt + \sigma dB_t^H,\ X_0 = 0. \tag{1.2}$$

The key difference between (1.1) and (1.2) is that $\mu$ is assumed to be zero and known in (1.2) while $\mu$ is unknown in (1.1). A small difference between (1.1) and (1.2) is that $X_0 = 0$ in (1.2) while $X_0$ may not be zero in (1.1). The order of the initial condition will be assumed when we develop the asymptotic theory.

fOU is closely related to the following discrete time model

$$y_t = \left(1 + \frac{\kappa}{T}\right)y_{t-1} + u_t,\ (1-L)^{H-1/2}u_t = \varepsilon_t,\ y_0 = 0,\ (t = 1, ..., T), \tag{1.3}$$

where $L$ is the lag operator, $\varepsilon_t \sim i.i.d.(0, \sigma^2)$. When $H = 1/2$, $u_t = \varepsilon_t$, $y_t$ follows a standard AR(1) model with an i.i.d. error term. When $1/2 < H < 1$, $u_t$ is a stationary long memory process given by

$$u_t = (1 - L)^{-(H-1/2)} \varepsilon_t = \sum_{j=0}^{\infty} \frac{\Gamma(j + H - 1/2)}{\Gamma(H - 1/2)\Gamma(j + 1)} \varepsilon_{t-j},$$

where $\Gamma(x)$ is the gamma function. Davydov (1970) and (29) related the process in (1.3) to that in (1.2) by showing the following functional limit theorem,

$$\frac{\delta_H \Gamma(H + 1/2)}{\sigma T^H} y_{[Ts]} \Rightarrow X_s \, , \forall 0 \le s \le 1.$$

where $[z]$ denotes the smallest integer greater than or equal to $z$, $\delta_H = \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}}$(see also (31) and (33)). If $\kappa = 0$, $y_t$ has a unit root; if $\kappa > 0$, $y_t$ is asymptotically stationary; if $\kappa < 0$, $y_t$ has an explosive root.

Depending on the sign of $\kappa$, alternative estimation methods have been proposed in the literature to estimate $\kappa$ in fOU and the asymptotic theory for these estimators have been obtained. When $\kappa > 0$, (20; 35; 32) studied the maximum likelihood (ML) estimator; (18) studied the least squares (LS) estimator; (31) studied the minimum contrast (MC) estimator; (18) introduced and studied an estimator based on the ergodic property of $X_t$. When $\kappa < 0$, two estimators have been studied, namely, the ML estimator ((33)) and the LS estimator ((3; 15)). When $\kappa = 0$ , the ML method and the MC method were considered in (20; 31). (28) is a textbook treatment of alternative methods and the asymptotic theory for estimating fOU.

In almost all empirically relevant cases, the parameter, $\mu$, in the drift function of Model (1.1) is unknown. Thus, it is important to estimate both $\kappa$ and $\mu$. In this paper, we consider the problem of estimating both $\kappa$ and $\mu$ in fVm based on a continuous record of observations over the period of $[0, T]$ with a known Hurst parameter $H \in [\frac{1}{2}, 1)$. As in fOU, the asymptotic theory for $\kappa$ critically depends on the sign of $\kappa$, namely whether $\kappa > 0$, $\kappa < 0$ or $\kappa = 0$. When $\kappa > 0$, two estimators are considered, i.e., the LS estimator and the estimator of Hu and Nualart (2010). The estimator of Hu and Nualart does not contain any stochastic integral and hence is simpler to calculate. Our results suggest that, unless $H = 1/2$, the estimator of Hu and Nualart is asymptotically more efficient than that of LS. The relative asymptotic efficiency increases with $H$. When $\kappa < 0$ or $\kappa = 0$, the LS estimator is considered. Strong consistency and asymptotic distributions are established for both $\kappa$ and $\mu$ in all three cases. The proof is based on the Malliavin calculus, the Young integral and the Stratonovich integral for fractional stochastic integrals (**Weilin, I thought we are going to use either Ito-Skorohod integral or Young integral to interpret the stochastic integral in the paper**). To the best of our knowledge, this is the first paper in the literature where fVm is estimated and the asymptotic theory is developed.

The rest of the paper is organized as follows. Section 2 contains some basic facts about fBm and introduces the LS method and the method of Hu and Nualart for estimating the two parameters in the drift function of fVm. In Section 3, we establish consistency and the asymptotic distributions for $\kappa$ and $\mu$ in all three cases. Section 4 contains some concluding remarks and gives directions of further research. All the proofs are collected in the Appendix.

We use the following notations throughout the paper: $\xrightarrow{p}$, $\xrightarrow{a.s.}$, $\xrightarrow{\mathcal{L}}$, $\Rightarrow$, and $\overset{d}{=}$ denote convergence in probability, convergence almost surely, convergence in distribution, weak convergence, and equivalence in distribution, respectively, as $T \to \infty$.

# 2 The Estimation Methods

Before introducing our estimation techniques, we first state some basic facts about fBm. For a more complete treatment on the subject, see (25; 5; 24) and the references therein.

The fBm with the Hurst parameter $H \in (0,1)$, $B_t^H$ for $t \in \mathbb{R}$, is a zero mean Gaussian process with covariance

$$\mathbb{E}(B_t^H B_s^H) = R_H(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) . \tag{2.1}$$

This covariance function implies that the fBm is self-similar with the self-similarity parameter $H$, that is,

$$B_{\lambda t}^H \overset{d}{=} \lambda^H B_t^H . \tag{2.2}$$

For $t > 0$, (23) presented the following integral representation for $B_t^H$ (see also in (13)):

$$B_t^H = \frac{1}{c_H}\left\{\int_{-\infty}^0 \left[(t-u)^{H-1/2} - (-u)^{H-1/2}\right] dW_u + \int_0^t (t-u)^{H-1/2} dW_u\right\}, \tag{2.3}$$

where $W_t$ is a standard Brownian motion, $c_H = \left[\frac{1}{2H} + \int_0^\infty \left((1+s)^{H-1/2} - s^{H-1/2}\right)^2 ds\right]^{1/2}$. If $H = 1/2$, $B_t^H$ becomes a standard Brownian motion $W_t$. If $0 < H < 1/2$, $B_t^H$ is negatively correlated. For $1/2 < H < 1$, $B_t^H$ has the long-range dependence in the sense that if $r(n) = \mathbb{E}(B_1^H(B_{n+1}^H - B_n^H))$, then $\sum_{n=1}^\infty r(n) = \infty$. In this case, $B_t^H$ is referred to as a persistent fBm, since the positive (negative) increments are likely to be followed by positive (negative) increments. Given that the long-range dependence is empirically found in many financial time series, the fVm with $H \in [1/2, 1)$ is the focus of the present paper. To estimate $\kappa$ and $\mu$ in fVm, we assume that one observes the whole trajectory of $X_t$ for $t \in [0,T]$. The asymptotic theory is developed by assuming $T \to \infty$.

Motivated by the work of (18; 3; 15), we denote the LS estimator of $\kappa$ and $\mu$ to be the minimizer of the following quadratic function

$$L(\kappa, \mu) = \int_0^T \left(\dot{X}_t - \kappa(\mu - X_t)\right)^2 dt\,, \tag{2.4}$$

where $\dot{X}_t$ denotes the differentiation of $X_t$ with respect to $t$, although $\int_0^T \dot{X}_t^2 dt$ does not exist. Consequently, we obtain the following analytical expression for the LS estimator of $\kappa$ and $\mu$ (denoted by $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$, respectively),

$$\hat{\kappa}_{LS} = \frac{(X_T - x) \int_0^T X_t dt - T \int_0^T X_t dX_t}{T \int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}, \tag{2.5}$$

$$\hat{\mu}_{LS} = \frac{(X_T - x) \int_0^T X_t^2 dt - \int_0^T X_t dX_t \int_0^T X_t dt}{(X_T - x) \int_0^T X_t dt - T \int_0^T X_t dX_t}. \tag{2.6}$$

When $H = 1/2$, it is well-known that we can interpret the stochastic integral $\int_0^T X_t dX_t$ as the Itô integral. When $H \in (\frac{1}{2}, 1)$, $X_t$ is no longer a semimartingale. In this case, for $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ to consistently estimate $\kappa$ and $\mu$, we have to interpret the stochastic integral $\int_0^T X_t dX_t$ carefully. In fact, we interpret it differently when the sign of $\kappa$ is different. If $\kappa > 0$, we interpret it as an Itô-Skorohod integral; if $\kappa < 0$, we interpret it as a Young integral; if $\kappa = 0$, we can interpret it either as an Itô-Skorohod integral or as a Young integral. The asymptotic distributions of $\hat{\kappa}_{LS}$ are different in these three cases.

If $\kappa > 0$, we can consider an alternative estimator of $\kappa$ and $\mu$ (denoting them by $\hat{\kappa}_{HN}$ and $\hat{\mu}_{HN}$, respectively). This estimator is motivated from (18) where the stationary and ergodic properties of a process that is closely related to $X_t$ were used to construct a new estimator for $\kappa$ in the fOU model. To fix the idea, the strong solution of fVm in (1.1) is given by

$$X_t = \mu + (X_0 - \mu) \exp(-\kappa t) + \sigma \int_{-\infty}^t e^{-\kappa(t-s)} dB_s^H, \tag{2.7}$$

which leads to the following discrete time representation (**Weilin, could you please check if the following equation is correct for fBm?**)

$$X_t = \mu + e^{-\kappa} (X_{t-1} - \mu) + \sigma \int_0^1 e^{-\kappa(1-s)} dB_{t+s}^H, \tag{2.8}$$

By the ergodic theorem, as $T \to \infty$, $\frac{1}{T} \int_0^T X_t dt \overset{a.s.}{\to} \mu$. So an alternative estimator of $\mu$ is

$$\hat{\mu}_{HN} = \frac{1}{T} \int_0^T X_t dt. \tag{2.9}$$

Moreover, following (18), we can show that when $\kappa > 0$,

$$T \int_0^T X_t^2 dt - \left( \int_0^T X_t dt \right)^2 \overset{a.s.}{\to} T^2 \sigma^2 \kappa^{-2H} H \Gamma(2H).$$

Hence, an alternative estimator of $\kappa$ is

$$\hat{\kappa}_{HN} = \left( \frac{T \int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}{T^2 \sigma^2 H \Gamma(2H)} \right)^{-\frac{1}{2H}}. \tag{2.10}$$

Compared with the LS estimators in (2.5) and (2.6) which involve stochastic integral $\int_0^T X_t dB_t^H$, the alternative estimators in (2.9) and (2.10) do not contain any stochastic integral. Hence, they are conceptually easier to understand and numerically easier to compute than the LS estimators.

# 3 Asymptotic Theory for $\kappa$ and $\mu$

In the case of Brownian motion- or Levy process-driven Vasicek models, it is known that the asymptotic theory for $\kappa$ depends on the sign of $\kappa$ (see, (39)). In the case of fVm, we show below that the asymptotic theory for $\kappa$ continues to depend on the sign of $\kappa$.

## 3.1 Asymptotic theory when $\kappa > 0$

In the context of fVm in (1.1), we can represent the stochastic integral $\int_0^T X_t dX_t$ as

$$\int_0^T X_t dX_t = \kappa\mu \int_0^T X_t dt - \kappa \int_0^T X_t^2 dt + \sigma \int_0^T X_t dB_t^H .$$

When $H = 1/2$ , the stochastic integral $\int_0^T X_t dB_t^H = \int_0^T X_t dB_t$ and can be interpreted as the well-known Itô stochastic integral. It can be approximated by the forward Riemann sums. When $H > 1/2$, we interpret $\int_0^T X_t dB_t^H$ as the Itô-Skorohod stochastic integral. In this case, following (14), $\int_0^T X_t dB_t^H$ is approximated by the Riemann sums defined in terms of the Wick product, i.e.,

$$\int_0^T X_t dB_t^H = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} X_{t_i} \diamond \left( B_{t_{i+1}}^H - B_{t_i}^H \right) , \tag{3.1}$$

where $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of $[0, T]$ with $|\pi| = \max_{0\le i\le n-1} (t_{i+1} - t_i)$.

Unfortunately, this approximation is less useful for computing the stochastic integral because the Wick product cannot be calculated just from the values of $X_{t_i}$ and $B_{t_{i+1}}^H - B_{t_i}^H$. In other words, unless $H = 1/2$, there is no computable representation of the term $\int_0^T X_t dX_t$ given the observations $X_t, t \in [0, T]$.

Using the Itô-Skorohod integral for fBm and the Malliavin derivative for $X_t$, we can rewrite $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ as

$$\hat{\kappa}_{LS} = \frac{\frac{X_T - X_0}{T}\frac{1}{T}\int_0^T X_t dt - \left(\frac{1}{2T}X_T^2 - \frac{1}{2T}X_0^2 - \frac{\alpha_H\sigma^2}{T}\int_0^T\int_0^t s^{2H-2}e^{-\kappa s} ds dt\right)}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2}, \tag{3.2}$$

$$\hat{\mu}_{LS} = \frac{\frac{X_T - X_0}{T}\frac{1}{T}\int_0^T X_t^2 dt - \frac{1}{T}\int_0^T X_t dt\left(\frac{X_T^2}{2T} - \frac{X_0^2}{2T} - \frac{\alpha_H\sigma^2}{T}\int_0^T\int_0^t s^{2H-2}e^{-\kappa s} ds dt\right)}{\frac{X_T - X_0}{T}\frac{1}{T}\int_0^T X_t dt - \left(\frac{X_T^2}{2T} - \frac{X_0^2}{2T} - \frac{\alpha_H\sigma^2}{T}\int_0^T\int_0^t s^{2H-2}e^{-\kappa s} ds dt\right)}, \tag{3.3}$$

where $\alpha_H = H(2H-1)$. Clearly, $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ in (3.2) and (3.3) are easier to compute than those in (2.5) and (2.6).

Before we prove the consistency of $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$, we first obtain the consistency of $\hat{\mu}_{HN}$ and $\hat{\mu}_{HN}$ which follows directly from the ergodicity.

**Theorem 3.1.** *Let $H \in [\frac{1}{2}, 1)$, $X_0/\sqrt{T} = o_p(1)$, and $\kappa > 0$ in (1.1). Then as $T \to \infty$, $\hat{\kappa}_{HN} \overset{a.s.}{\to} \kappa$ and $\hat{\mu}_{HN} \overset{a.s.}{\to} \mu$.*

**Remark 3.1.** *The almost sure convergence of $\hat{\kappa}_{HN}$ in Theorem 3.1 extends that of (18) from fOU to fVm.*

**Remark 3.2.** *Applying the well-known result that $\frac{1}{T}\int_0^T \int_0^t s^{2H-2} e^{-\kappa s} ds dt \to \kappa^{1-2H}\Gamma(2H-1)$ to (3.2) and (3.3) and using Lemma 5.2 in (18), we can show that, as $T \to \infty$, $\hat{\kappa}_{LS} \overset{a.s.}{\to} \kappa$ and $\hat{\mu}_{LS} \overset{a.s.}{\to} \mu$ for $H \in [\frac{1}{2}, 1)$.*

To establish the asymptotic distributions for the two sets of estimators, we first consider $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$, and then use the asymptotic theory for $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ to develop the asymptotic theory for $\hat{\mu}_{HN}$ and $\hat{\mu}_{HN}$.

**Theorem 3.2.** *Let $H \in [\frac{1}{2}, 1)$, $X_0/\sqrt{T} = o_p(1)$, and $\kappa > 0$ in (1.1). Then as $T \to \infty$,*

$$\sqrt{T}\left(\hat{\kappa}_{LS} - \kappa\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \kappa C_H\right), \tag{3.4}$$

*where $C_H = (4H-1)\left(1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2H)\Gamma(2-2H)}\right)$.*

**Remark 3.3.** *A straightforward calculation shows that*

$$\begin{aligned} T^{1-H}\left(\hat{\mu}_{LS} - \mu\right) &= \frac{\frac{X_T - x}{T^H}\frac{1}{T}\int_0^T X_t^2 dt - \frac{1}{T}\int_0^T X_t dX_t \frac{1}{T^H}\int_0^T X_t dt}{\frac{X_T - x}{T}\frac{1}{T}\int_0^T X_t dt - \frac{1}{T}\int_0^T X_t dX_t} \\ &\quad - \frac{\mu\left(\frac{X_T - x}{T^H}\frac{1}{T}\int_0^T X_t dt - \frac{1}{T^H}\int_0^T X_t dX_t\right)}{\frac{X_T - x}{T}\frac{1}{T}\int_0^T X_t dt - \frac{1}{T}\int_0^T X_t dX_t}. \end{aligned}$$

*For $H \in [\frac{1}{2}, 1)$, we can easily obtain the following asymptotic distribution of $\hat{\mu}_{LS}$,*

$$T^{1-H}\left(\hat{\mu}_{LS} - \mu\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \frac{\sigma^2}{\kappa^2}\right) \quad \text{as } T \to \infty. \tag{3.5}$$

**Theorem 3.3.** *Let $H \in [\frac{1}{2}, 1)$, $X_0/\sqrt{T} = o_p(1)$, and $\kappa > 0$ in (1.1). Then as $T \to \infty$,*

$$T^{1-H}\left(\hat{\mu}_{HN} - \mu\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \frac{\sigma^2}{\kappa^2}\right). \tag{3.6}$$

*Moreover, if $H \in [\frac{1}{2}, \frac{3}{4})$, $X_0/\sqrt{T} = o_p(1)$, and $\kappa > 0$ in (1.1), then*

$$\sqrt{T}\left(\hat{\kappa}_{HN} - \kappa\right) \overset{\mathcal{L}}{\to} \mathcal{N}\left(0, \kappa\rho_H\right), \tag{3.7}$$

*where $\rho_H = \frac{4H-1}{4H^2}\left(1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2H)\Gamma(2-2H)}\right) = \frac{C_H}{4H^2}$.*

**Remark 3.4.** *When comparing the two sets of the asymptotic theory for $\kappa$, we can draw a few conclusions. First, the rate of convergence for $\hat{\kappa}_{HN}$ is the same as that for $\hat{\kappa}_{LS}$ which is $\sqrt{T}$ and independent on $H$. Second, the two asymptotic variances depend on $H$. When $H = 1/2$, the two estimators have the same asymptotic variance which is $2\kappa$. In this case, the asymptotic distribution is identical to that in (16), i.e., $\mathcal{N}(0, 2\kappa)$. When $1/2 < H < 1$, $4H^2 > 1$ and hence the asymptotic variance of $\hat{\kappa}_{HN}$ is smaller than that of $\hat{\kappa}_{LS}$, suggesting that the method of Hu and Nualart can estimate $\kappa$ more efficiently. Third, the asymptotic distribution of $\hat{\kappa}_{LS}$ and $\hat{\kappa}_{HN}$ is the same as that in fOU respectively; see p. 1034 and p. 1037 in (18).*

**Remark 3.5.** *The two sets of asymptotic theory for $\mu$ are identical and the rate of convergence is $T^{1-H}$. These two features are different from $\kappa$.*

**Remark 3.6.** *The asymptotic variance of $\hat{\kappa}_{HN}$ and $\hat{\kappa}_{LS}$ depends on $H$. Figure 1 plots $\rho_H$ and $C_H$ as a function of $H$. Obviously, both $\rho_H$ and $C_H$ monotonically increase in $H$ over the interval $[\frac{1}{2}, \frac{3}{4})$. They reach the minimum value of 2 when $H = 1/2$. As $H \to 3/4$, both diverge to $\infty$. Hence, both $\rho_H$ and $C_H$ have a singularity at $H = 3/4$. Since $\rho_H$ diverges faster than $C_H$, the relative asymptotic efficiency of $\hat{\kappa}_{HN}$ to $\hat{\kappa}_{LS}$ increases in $H$.*

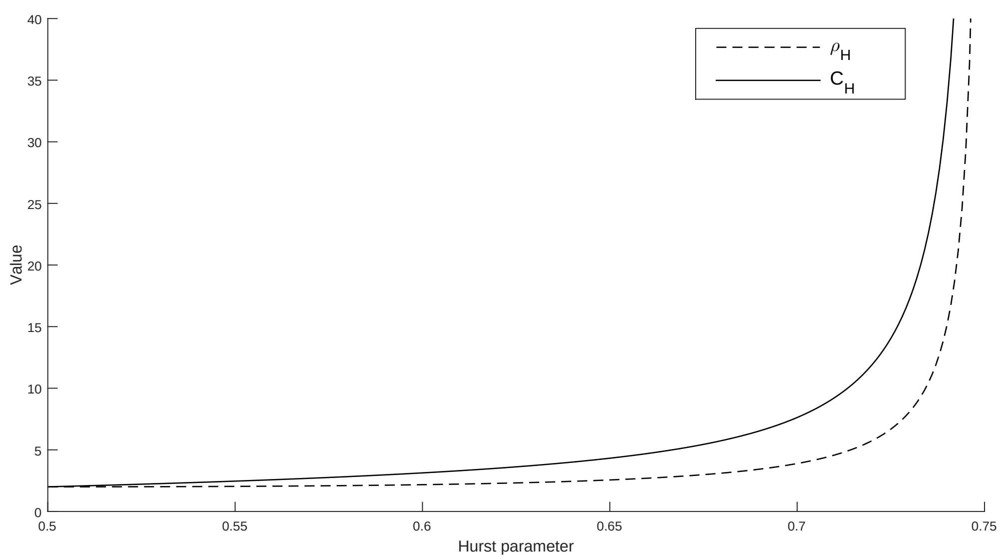


FIGURE 1. Plot of $\rho_H$ and $C_\rho$

**Remark 3.7.** *It seems challenging to obtain the asymptotic distribution for estimators of $\kappa$ when $H > 3/4$. It is expected that the asymptotic normality cannot hold when $H > 3/4$. This conjecture arises because of the results obtained in (6), where it is shown that the asymptotic distribution of the empirical quadratic variations of fBm is normal if $H < 3/4$ but is non-normal if $H > 3/4$.*

## 3.2 Asymptotic theory when $\kappa < 0$

When $\kappa < 0$, the model is explosive. Inspired by (3), we interpret the stochastic integral $\int_0^T X_t dX_t$ in (2.5) and (2.6) as the Young integral (see (42)) to get the consistent estimation of $\kappa$ and $\mu$. In this case, $\int_0^T X_t dX_t = (X_T^2 - X_0^2)/2$.

Applying the Young integral to (2.5) and (2.6), we can rewrite $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ as

$$\begin{aligned}
\hat{\kappa}_{LS} &= \frac{(X_T - X_0)\int_0^T X_t dt - \frac{T}{2}\left(X_T^2 - X_0^2\right)}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \\
&= \frac{\frac{X_T}{T}e^{\kappa T}e^{\kappa T}\int_0^T X_t dt - \frac{X_0}{T}e^{\kappa T}e^{\kappa T}\int_0^T X_t dt - \frac{1}{2}X_T^2 e^{2\kappa T} + \frac{1}{2}X_0^2 e^{2\kappa T}}{e^{2\kappa T}\int_0^T X_t^2 dt - e^{2\kappa T}\frac{1}{T}\left(\int_0^T X_t dt\right)^2}, \qquad (3.8) \\
\hat{\mu}_{LS} &= \frac{(X_T - X_0)\int_0^T X_t^2 dt - \frac{X_T^2 - X_0^2}{2}\int_0^T X_t dt}{(X_T - X_0)\int_0^T X_t dt - T\frac{X_T^2 - X_0^2}{2}} \\
&= \frac{\frac{e^{\kappa T}}{T}\int_0^T X_t^2 dt - \frac{X_T + X_0}{2T}e^{\kappa T}\int_0^T X_t dt}{\frac{e^{\kappa T}}{T}\int_0^T X_t dt - \frac{X_T + X_0}{2}e^{\kappa T}} \qquad (3.9)
\end{aligned}$$

Before considering the strong consistency of $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$, we first introduce a lemma, which will be used to prove the strong consistency.

**Lemma 3.1.** *Let $H \in [\frac{1}{2}, 1)$, $X_0 = O_p(1)$, and $\kappa < 0$ in (1.1) (**Weilin, although I assume $X_0 = O_p(1)$ in this subsection, we may be able to allow for a larger initial condition here**). Then, as $T \to \infty$, we have*

$$\frac{e^{\kappa T}}{T^H}\int_0^T X_t dB_t^H \xrightarrow{a.s.} 0\,.$$

**Theorem 3.4.** *Let $H \in [\frac{1}{2}, 1)$, $X_0 = O_p(1)$, and $\kappa < 0$ in (1.1). Then, as $T \to \infty$, $\hat{\kappa}_{LS} \xrightarrow{a.s.} \kappa$ and $\hat{\mu}_{LS} \xrightarrow{a.s.} \mu$.*

The asymptotic distributions for $\hat{\kappa}_{LS}$ and $\hat{\mu}_{LS}$ is developed in the following Theorem.

**Theorem 3.5.** *Let $H \in [\frac{1}{2}, 1)$, $X_0 = O_p(1)$, and $\kappa < 0$ in (1.1). Then as $T \to \infty$,*

$$\frac{e^{-\kappa T}}{2\kappa}\left(\hat{\kappa}_{LS} - \kappa\right) \xrightarrow{\mathcal{L}} \frac{\sigma\frac{\sqrt{H\Gamma(2H)}}{|\kappa|^H}\nu}{X_0 - \mu + \sigma\frac{\sqrt{H\Gamma(2H)}}{|\kappa|^H}\omega}, \qquad (3.10)$$

*where $\nu$ and $\omega$ are two independent standard normal variates. Moreover, as $T \to \infty$,*

$$T^{1-H}\left(\hat{\mu}_{LS} - \mu\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\sigma^2}{\kappa^2}\right). \qquad (3.11)$$

**Remark 3.8.** *In (3.10), if we set $X_0 = \mu$, the limiting distribution of $\frac{e^{-\kappa T}}{2\kappa}(\hat{\kappa}_{LS} - \kappa)$ becomes $\nu/\omega$ which is a standard Cauchy variate. This limiting distribution is the same as that in fOU (see, e.g., (3; 15)) and that in the Vasicek model driven by the standard Brownian motion (see, e.g., (16)). Moreover, the asymptotic theory in (3.10) is similar to that in the explosive discrete time and continuous time models when discrete-sampled data are available (see, e.g., (40***?*** ; 39; 38)).*

**Remark 3.9.** *In the context of fOU, (3) showed that the LS estimator of $\kappa$ is consistent and derived the asymptotic Cauchy distribution. Our result not only extends their result on $\kappa$ to a more general model with an unknown $\mu$ and a more general initial condition, but also covers the asymptotic theory for $\mu$. The asymptotic distribution of $\hat{\mu}_{LS}$ is normal with the rate of convergence being $T^{1-H}$ and the variance being $\sigma^2/\kappa^2$ when $\kappa < 0$. This asymptotic distribution is the same as that of $\hat{\mu}_{LS}$ and $\hat{\mu}_{HN}$ when $\kappa > 0$ as shown in (3.5) and (3.6).*

## 3.3 Asymptotic theory when $\kappa = 0$

When $\kappa = 0$, fVm is null recurrent. In this case, we have

$$X_t = X_0 + \sigma B_t^H ,$$

and the parameter $\mu$ vanishes. By a simple calculation, we have

$$\begin{aligned}
\hat{\kappa}_{LS} &= \frac{\sigma B_t^H \int_0^T \left(x + \sigma B_t^H\right) dt - T\sigma \int_0^T \left(x + \sigma B_t^H\right) dB_t^H}{T \int_0^T \left(x + \sigma B_t^H\right)^2 dt - \left(\int_0^T \left(x + \sigma B_t^H\right) dt\right)^2} \qquad (3.12) \\
&= \frac{B_T^H \int_0^T B_t^H dt - T \int_0^T B_t^H dB_t^H}{T \int_0^T (B_t^H)^2 dt - \left(\int_0^T B_t^H dt\right)^2} .
\end{aligned}$$

Interestingly, for $\hat{\kappa}_{LS}$ to consistently estimate $\kappa$, the stochastic integral $\int_0^T B_t^H dB_t^H$ can be interpreted either as the Itô-Skorohod integral or as the Young integral.

If we interpret $\int_0^T B_t^H dB_t^H$ as the Itô-Skorohod integral, we have

$$\hat{\kappa}_{LS} = \frac{B_T^H \int_0^T B_t^H dt - \frac{T}{2}\left(\left(B_T^H\right)^2 - T^{2H}\right)}{T \int_0^T (B_t^H)^2 dt - \left(\int_0^T B_t^H dt\right)^2} .$$

By the law of the iterated logarithm for fBm, we get $\hat{\kappa}_{LS} \overset{a.s.}{\to} 0$.

If we interpret $\int_0^T B_t^H dB_t^H$ as the Young integral, we have

$$\hat{\kappa}_{LS} = \frac{B_T^H \int_0^T B_t^H dt - \frac{T}{2}\left(B_T^H\right)^2}{T \int_0^T (B_t^H)^2 dt - \left(\int_0^T B_t^H dt\right)^2} .$$

Again by the law of the iterated logarithm for fBm, we get $\hat{\kappa}_{LS} \stackrel{a.s.}{\to} 0$.

Using the results above and the scaling properties of fBm, we develop the following asymptotic distribution for $\hat{\kappa}_{LS}$.

**Theorem 3.6.** *Let $H \in [\frac{1}{2}, 1)$, $X_0 = O_p(1)$, and $\kappa = 0$ in (1.1). Then as $T \to \infty$,*

$$T\hat{\kappa}_{LS} \stackrel{\mathcal{L}}{\to} \frac{B_1^H \int_0^1 B_u^H du - \int_0^1 B_u^H dB_u^H}{\int_0^1 (B_u^H)^2 du - \left(\int_0^1 B_u^H du\right)^2}. \tag{3.13}$$

**Remark 3.10.** *This limiting distribution is neither a normal variate nor a mixture of normals. In addition, the distribution depends on $H$. If $H = 1/2$ the limiting distribution becomes a Dickey-Fuller-Phillips type of distribution (see, e.g. (27)) which has been widely used for testing unit root in autoregression with an intercept included. (**?** )) derived the limiting distribution of the LS estimator of $\kappa$ in fOU when $\kappa = 0$. The limiting distribution is another Dickey-Fuller-Phillips type of distribution (see, e.g. (Phillips1987)) and corresponds to that in autoregression with an intercept.*

# 4 Concluding Remarks and Future Directions

Models with a long-range dependence are growing their popularity due to their empirical success in practice. In the continuous time setting, the long-range dependence can be modelled with the help of fBm when the Hurst parameter is greater than one half. Consequently, statistical inference for stochastic models driven by fBm is important. This paper considers the Vasicek model driven by fBm and deals with the estimation problem of the two parameters in the drift function in fVm and their asymptotic theory when a continuous record of observations is available.

As the time span goes to infinity, it is shown that the LS estimators of $\mu$ and $\kappa$ are strongly consistent regardless of the sign of the persistent parameter $\kappa$. Moreover, the asymptotic distribution of the LS estimator of $\mu$ is asymptotically normal regardless of the sign of $\kappa$. However, the asymptotic distribution of the LS estimator of $\kappa$ critically depends on the sign of $\kappa$. In particular, when $\kappa > 0$ and $H \in [\frac{1}{2}, \frac{3}{4})$, we have shown that the asymptotic distribution of the LS estimator of $\kappa$ is normal with the rate of convergence being the square root of time span. The asymptotic variance depends on $H$ which monotonically increases in $H$. When $\kappa < 0$, it is shown that the limiting distribution is a Cauchy-type with the rate of convergence being $e^{-\kappa T}$. If $\mu$ is the same as the initial condition, it becomes the standard Cauchy distribution. When $\kappa = 0$, the asymptotic distribution is neither normal nor a mixture of normals, but a Dickey-Fuller type of distribution. The rate of convergence is $T$. When $\kappa > 0$, we also consider an alternative estimation technique by exploiting the ergodic property of fVm. The asymptotic theory for the alternative estimator is established.

This study also suggests several important directions for future research. First, what are the asymptotic properties of the ML estimators for $\kappa$ and $\mu$? Given that the model is fully parametrically specified, one may wish to estimate fVm using ML. Based on the fractional version of Girsanov's theorem, one can obtain the Radon-Nikodym derivative and the log-likelihood function. Consequently, the ML estimators can be obtained. The asymptotic properties of ML estimators can be derived by using the Laplace transform.

Second, the present study assumes a continuous record is available for parameter estimation. This assumption is too strong in almost all empirically relevant cases. How to estimate parameters in fVm from discrete time observations and how to obtain the asymptotic theory are open questions.

Third, when $\kappa > 0$ and $H \geq \frac{3}{4}$, the asymptotic distribution remains unknown for the LS estimator. When $H = 3/4$, the fourth moment Berry-Esseen bound is perhaps needed to obtain the asymptotic normality for $\hat{\kappa}_{LS}$. When $H > 3/4$, it may be expected that the asymptotic normality cannot hold any more for $\hat{\kappa}_{LS}$ due to a result obtained in (6).

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# APPENDIX: Proof of Theoretic Results

## A.1. Proof of Theorem 3.1

We first consider the strong consistency of $\hat{\mu}_{HN}$. Based on the assumption $X_0 = X_0$, we can obtain that the solution of (1.1)

$$X_t = \left(1 - e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} dB_s^H \,. \tag{A.1}$$

For $t \geq 0$, we define

$$Y_t = \sigma \int_{-\infty}^t e^{-\kappa(t-s)} dB_s^H \,. \tag{A.2}$$

Since $\kappa > 0$, $(Y_t, t \geq 0)$ is Gaussian, stationary and ergodic. Then, using the ergodic theorem and the fact $\mathbb{E}\left[Y_0\right] = 0$, we obtain

$$\frac{1}{T}\int_0^T Y_t dt \xrightarrow{a.s.} \mathbb{E}\left(Y_0\right) = 0 \,. \tag{A.3}$$

Combining (A.1) and (A.2), we can rewrite $Y_t$ as,

$$Y_t = X_t + \left(e^{-\kappa t} - 1\right)\mu - X_0 e^{-\kappa t} + \sigma \int_{-\infty}^0 e^{-\kappa(t-s)} dB_s^H \,. \tag{A.4}$$

Hence,

$$\begin{aligned} \frac{1}{T}\int_0^T Y_t dt &= \frac{1}{T}\int_0^T \left[X_t + \mu\left(e^{-\kappa t} - 1\right) - X_0 e^{-\kappa t} + e^{-\kappa t}\left(\sigma \int_{-\infty}^0 e^{\kappa s} dB_s^H\right)\right] dt \\ &= \frac{1}{T}\int_0^T X_t dt + \frac{\mu}{T}\int_0^T \left(e^{-\kappa t} - 1\right) dt - \frac{X_0}{T}\int_0^T e^{-\kappa t} dt \\ &\quad + \frac{\sigma}{T}\int_0^T e^{-\kappa t}\left(\int_{-\infty}^0 e^{\kappa s} dB_s^H\right) dt \,. \end{aligned} \tag{A.5}$$

For the second term in (A.5), it is obvious that

$$\frac{\mu}{T}\int_0^T \left(e^{-\kappa t}-1\right) dt \overset{a.s.}{\to} -\mu\,.$$

Since $X_0 = o_p(\sqrt{T})$, we obtain

$$\frac{X_0}{T}\int_0^T e^{-\kappa t} dt \overset{a.s.}{\to} 0\,.$$

Using an argument similar to that in Lemma 5.1 of (18), we have

$$\mathbb{E}\left[\int_{-\infty}^0 e^{\kappa s} dB_s^H\right]^2 = \kappa^{-2H} H\Gamma(2H)\,. \tag{A.6}$$

Hence, $\int_{-\infty}^0 e^{\kappa s} dB_s^H$ has the limiting (normal) distribution of $\int_0^T e^{-\kappa(T-s)} dB_s^H$. Moreover, a standard calculation yields

$$\int_0^T e^{-\kappa t} dt \overset{a.s.}{\to} \frac{1}{\kappa}\,. \tag{A.7}$$

It is now necessary to investigate the almost sure asymptotic behavior of the last term in (A.5). Denote $F_T = \frac{\sigma}{\sqrt{T}}\int_0^T e^{-\kappa t}\left(\int_{-\infty}^0 e^{\kappa s} dB_s^H\right) dt$. From (A.6) and (A.7), we see that $\sup_T \mathbb{E}\left[|F_T^2|\right] < \infty$ and $\sup_T \mathbb{E}\left[|F_T^4|\right] < \infty$. For any fixed $\varepsilon > 0$, it follows from Chebyshev's inequality that

$$\mathbb{P}\left(\left|\frac{\sigma}{T}\int_0^T e^{-\kappa t}\left(\int_{-\infty}^0 e^{\kappa s} dB_s^H\right) dt\right| > \varepsilon\right) = \mathbb{P}\left(|F_T| > \sqrt{T}\varepsilon\right) \le \frac{81}{T^2\varepsilon^4}\mathbb{E}\left[\left|F_T^2\right|\right]^2\,.$$

Then, the Borel-Cantelli lemma implies that

$$\frac{\sigma}{T}\int_0^T e^{-\kappa t}\left(\int_{-\infty}^0 e^{\kappa s} dB_s^H\right) dt \overset{a.s.}{\to} 0\,. \tag{A.8}$$

Plugging all these convergency results to (A.5), we obtain

$$\hat{\mu}_{HN} = \frac{1}{T}\int_0^T X_t dt \overset{a.s.}{\to} \mu\,. \tag{A.9}$$

To establish the strong consistency of $\hat{\kappa}_{HN}$ defined in (2.10), we need to consider the strong consistency of $\frac{1}{T}\int_0^T X_t^2 dt$. From the expression of $Y_t$ in (A.4), we obtain

$$\begin{aligned}
\frac{1}{T}\int_0^T Y_t^2 dt &= \frac{1}{T}\int_0^T \left[X_t + \mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t} + e^{-\kappa t}\left(\sigma\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)\right]^2 dt \qquad \text{(A.10)}\\
&= \frac{1}{T}\int_0^T \left[X_t + \mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t}\right]^2 dt + \frac{1}{T}\int_0^T \left[e^{-\kappa t}\left(\sigma\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)\right]^2 dt\\
&\quad + \frac{2}{T}\int_0^T \left[X_t + \mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t}\right]\left[e^{-\kappa t}\left(\sigma\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)\right] dt\\
&= \frac{1}{T}\int_0^T \left[\mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t}\right]^2 dt + \frac{2}{T}\int_0^T X_t\left[\mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t}\right] dt\\
&\quad + \frac{1}{T}\int_0^T X_t^2 dt + \frac{1}{T}\int_0^T \left[e^{-\kappa t}\left(\sigma\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)\right]^2 dt\\
&\quad + \frac{2}{T}\int_0^T \left[X_t + \mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t}\right]\left[e^{-\kappa t}\left(\sigma\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)\right] dt\,.
\end{aligned}$$

By (A.8) and Lemma 3.3 in (18), it is not hard to see that

$$\frac{\sigma^2}{T}\int_0^T \left[\int_0^t e^{-\kappa(t-s)} dB_s^H + e^{-\kappa t}\left(\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)\right]^2 dt - \frac{\sigma^2}{T}\int_0^T \left[\int_0^t e^{-\kappa(t-s)} dB_s^H\right]^2 dt \xrightarrow{a.s.} 0\,.$$

Combining the above result and (A.8), we deduce that

$$\frac{2}{T}\int_0^T \left[\sigma\int_0^t e^{-\kappa(t-s)} dB_s^H\right]\left[e^{-\kappa t}\left(\sigma\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)\right] dt \xrightarrow{a.s.} 0\,.$$

Using (A.1) and (**??**), we obtain

$$\frac{2}{T}\int_0^T \left[X_t + \mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t}\right]\left[e^{-\kappa t}\left(\sigma\int_{-\infty}^0 e^{\kappa s} dB_s^H\right)\right] dt \xrightarrow{a.s.} 0\,. \qquad \text{(A.11)}$$

A standard calculation yields

$$\frac{2}{T}\int_0^T X_t\left[\mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t}\right] dt \xrightarrow{a.s.} -2\mu^2\,, \qquad \text{(A.12)}$$

$$\frac{1}{T}\int_0^T \left[\mu\left(e^{-\kappa t}-1\right) - X_0 e^{-\kappa t}\right]^2 dt \xrightarrow{a.s.} \mu^2\,. \qquad \text{(A.13)}$$

By (A.10) - (A.13) and the ergodic theorem, we obtain

$$\frac{1}{T}\int_0^T X_t^2 dt \xrightarrow{a.s.} \mathbb{E}\left(Y_0^2\right) + \mu^2\,. \qquad \text{(A.14)}$$

Moreover, it is well-known that (see, e.g., Lemma 5.1 of (18))

$$\mathbb{E}(Y_0^2) = \alpha_H \sigma^2 \int_0^\infty \int_0^\infty e^{-\kappa(s+u)} |u-s|^{2H-2} duds = \sigma^2 \kappa^{-2H} H\Gamma(2H) \,. \tag{A.15}$$

Combining (A.14) and (A.15), we have

$$\frac{1}{T}\int_0^T X_t^2 dt \overset{a.s.}{\to} \sigma^2\kappa^{-2H} H\Gamma(2H) + \mu^2 \,. \tag{A.16}$$

By (A.9), (A.16) and the arithmetic rule of convergence, we obtain the almost sure convergence of $\hat{\kappa}_{HN}$ defined in (2.10), i.e., $\hat{\kappa}_{HN} \overset{a.s.}{\to} \kappa$.

## A.2. Proof of Theorem 3.2

Based on (2.5), (1.1) and (A.1), we can rewrite $\kappa_T$ as

$$\begin{aligned}
\hat{\kappa}_{LS} &= \frac{(X_T - x)\int_0^T X_t dt - \kappa\mu T\int_0^T X_t dt + \kappa T \int_0^T X_t^2 dt - \sigma T \int_0^T X_t dB_t^H}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \\
&= \kappa + \frac{(X_T - x)\int_0^T X_t dt - \kappa\mu T\int_0^T X_t dt - \sigma T \int_0^T X_t dB_t^H + \kappa\left(\int_0^T X_t dt\right)^2}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \\
&= \kappa - \frac{\sigma T\int_0^T X_t dB_t^H}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} + \frac{\left(X_T - X_0 - \kappa\mu T + \kappa\int_0^T X_t dt\right)\int_0^T X_t dt}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \\
&= \kappa - \frac{\sigma T\int_0^T \left(\left(1-e^{-\kappa t}\right)\mu + X_0 e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)} dB_s^H\right) dB_t^H}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \\
&\quad + \frac{\left(X_T - x + \kappa\int_0^T \left(X_0 e^{-\kappa t} - \mu e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)} dB_s^H\right) dt\right)\int_0^T X_t dt}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \,.
\end{aligned}$$

As a consequence, we have the following decomposition

$$\begin{aligned}
&\sqrt{T}\left(\hat{\kappa}_{LS} - \kappa\right) \qquad\qquad \text{(A.17)} \\
&= -\frac{\sigma\left(\mu\frac{B_T^H}{\sqrt{T}} + \frac{x-\mu}{\sqrt{T}}\int_0^T e^{-\kappa t} dB_t^H + \frac{\sigma}{\sqrt{T}}\int_0^T\int_0^t e^{-\kappa(t-s)} dB_s^H dB_t^H\right)}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2} \\
&\quad + \frac{\left(\frac{X_T - x}{\sqrt{T}} + \frac{\kappa(x-\mu)}{\sqrt{T}}\int_0^T e^{-\kappa t} dt - \frac{\sigma}{\sqrt{T}} e^{-\kappa T}\int_0^T e^{\kappa s} dB_s^H + \sigma\frac{B_T^H}{\sqrt{T}}\right)\frac{1}{T}\int_0^T X_t dt}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2} \\
&= I_1 + I_2 + I_3 \,,
\end{aligned}$$

where

$$
\begin{aligned}
I_1 &= \frac{\sigma\left(\frac{\mu-x}{\sqrt{T}}\int_0^T e^{-\kappa t}dB_t^H - \frac{\sigma}{\sqrt{T}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H\right)}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2}, \\
I_2 &= \frac{\left(\frac{X_T-x}{\sqrt{T}} + \frac{\kappa(x-\mu)}{\sqrt{T}}\int_0^T e^{-\kappa t}dt - \frac{\sigma}{\sqrt{T}}e^{-\kappa T}\int_0^T e^{\kappa s}dB_s^H\right)\frac{1}{T}\int_0^T X_t dt}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2}, \\
I_3 &= \frac{\left(-\mu\sigma + \frac{\sigma}{T}\int_0^T X_t dt\right)\frac{B_T^H}{\sqrt{T}}}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2}.
\end{aligned}
$$

First consider $I_1$. Using (A.15), we have

$$
\mathbb{E}\left[\left(\frac{\mu\sigma}{\sqrt{T}}\int_0^T e^{-\kappa t}dB_t^H\right)^2\right] = \frac{\mu^2\sigma^2}{T}\alpha_H\int_0^T\int_0^T e^{-\kappa(s+u)}\left|u-s\right|^{2H-2}duds \xrightarrow{a.s.} 0\,.
$$

This implies

$$
\frac{\mu\sigma}{\sqrt{T}}\int_0^T e^{-\kappa t}dB_t^H \xrightarrow{p} 0\,. \tag{A.18}
$$

Since $X_0 = o_p(\sqrt{T})$, we have

$$
\frac{X_0\sigma}{\sqrt{T}}\int_0^T e^{-\kappa t}dB_t^H \xrightarrow{p} 0\,. \tag{A.19}
$$

Furthermore, from Theorem 3.4 of (18), (A.9) and (A.16), we obtain

$$
\frac{-\frac{\sigma^2}{\sqrt{T}}\int_0^T\int_0^t\ e^{-\kappa(t-s)}dB_s^H dB_t^H}{\frac{1}{T}\int_0^T X_t^2 dt - \left(\frac{1}{T}\int_0^T X_t dt\right)^2} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \kappa C_H\right)\,, \tag{A.20}
$$

where $C_H = (4H-1)\left(1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2H)\Gamma(2-2H)}\right)$. Combining (A.18), (A.19), (A.20) and applying Slutsky's theorem, we have

$$
I_1 \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \kappa C_H\right)\,. \tag{A.21}
$$

Next, we consider $I_2$. From Lemma 5.2 and Eq. (3.8) in (18), we have

$$
\frac{X_T - X_0}{\sqrt{T}} \xrightarrow{a.s.} 0\,,\quad \frac{\sigma}{\sqrt{T}}e^{-\kappa T}\left(\int_0^T \sigma e^{\kappa s}dB_s^H\right) \xrightarrow{a.s.} 0\,. \tag{A.22}
$$

A straightforward calculation shows that

$$
\frac{\kappa\left(X_0 - \mu\right)}{\sqrt{T}}\int_0^T e^{-\kappa t}dt \xrightarrow{a.s.} 0\,. \tag{A.23}
$$

Combining (A.22), (A.23), (A.9) and (A.16), we have

$$I_2 \xrightarrow{a.s.} 0\,. \tag{A.24}$$

Finally, consider $I_3$. Based on (A.1), we have

$$\begin{aligned}
&\left(-\mu\sigma + \frac{\sigma}{T}\int_0^T X_t dt\right)\frac{B_T^H}{\sqrt{T}} \qquad\qquad (A.25)\\
&= \frac{\sigma}{T}\int_0^T \left((X_0-\mu)\,e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)}dB_s^H\right)dt \cdot \frac{B_T^H}{\sqrt{T}}\\
&= \left(\frac{\sigma\,(X_0-\mu)}{T^{\frac{3}{2}-H}}\int_0^T e^{-\kappa t}dt - \frac{\sigma^2}{\kappa T^{\frac{3}{2}-H}}e^{-\kappa T}\int_0^T e^{\kappa s}dB_s^H + \frac{\sigma^2}{\kappa}\frac{B_T^H}{T^{\frac{3}{2}-H}}\right)\frac{B_T^H}{T^H}\,.
\end{aligned}$$

It is easy to see that

$$\frac{\sigma\,(X_0-\mu)}{T^{\frac{3}{2}-H}}\int_0^T e^{-\kappa t}dt \xrightarrow{a.s.} 0\,. \tag{A.26}$$

From Lemma 5.2 and Eq. (3.8) in (18)), we obtain

$$\frac{\sigma^2}{\kappa T^{\frac{3}{2}-H}}e^{-\kappa T}\int_0^T e^{\kappa s}dB_s^H \xrightarrow{a.s.} 0\,. \tag{A.27}$$

Since $H \in [\frac{1}{2}, \frac{3}{4})$, we have

$$\mathbb{E}\left[\left(\frac{\sigma^2}{\kappa}\frac{B_T^H}{T^{\frac{3}{2}-H}}\right)^2\right] = \frac{\sigma^4}{\kappa^2}T^{4H-3}\,,$$

which implies

$$\frac{\sigma^2}{\kappa}\frac{B_T^H}{T^{\frac{3}{2}-H}} \xrightarrow{p} 0\,. \tag{A.28}$$

By (A.25) - (A.28), we obtain

$$I_3 \xrightarrow{p} 0\,. \tag{A.29}$$

Finally, by (A.17), (A.21), (A.24), (A.29) and Slutsky's theorem, we obtain the desired result in (3.4).

## A.3. Proof of Theorem 3.3

We first consider the asymptotic distribution of $\hat{\mu}_{HN}$. Using (A.1), we obtain

$$\begin{aligned}
T^{1-H}\left(\frac{1}{T}\int_0^T X_t dt - \mu\right) &= T^{1-H}\left[\frac{1}{T}\int_0^T \left((x-\mu)\,e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)}dB_s^H\right)dt\right] \qquad (A.30)\\
&= \frac{X_0-\mu}{T^H}\int_0^T e^{-\kappa t}dt + \frac{\sigma}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dt\,.
\end{aligned}$$

A simply calculation yields

$$\frac{X_0-\mu}{T^H}\int_0^T e^{-\kappa t}dt \xrightarrow{a.s.} 0\,. \tag{A.31}$$

Moreover, a standard calculation together with Fubini's stochastic theorem (see, e.g., (25)) yields

$$\begin{aligned}\frac{\sigma}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dt &= \frac{\sigma}{T^H}\int_0^T e^{\kappa s}\int_s^T e^{-\kappa t}dt dB_s^H \\ &= -\frac{\sigma}{\kappa T^H}\int_0^T e^{-\kappa(T-s)}dB_s^H + \frac{\sigma B_T^H}{\kappa T^H}\,.\end{aligned} \tag{A.32}$$

From Eq. (3.8) of (18), we know that

$$\frac{\sigma}{\kappa T^H}\int_0^T e^{-\kappa(T-s)}dB_s^H \xrightarrow{a.s.} 0\,. \tag{A.33}$$

It is well-known that

$$\frac{B_T^H}{T^H} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)\,. \tag{A.34}$$

By (A.32), (A.33), (A.34) and Slutsky's theorem, we have

$$\frac{\sigma}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dt \xrightarrow{\mathcal{L}} \mathcal{N}(0,\frac{\sigma^2}{\kappa^2})\,. \tag{A.35}$$

Combining (A.30), (A.31) and (A.35) and by Slutsky's theorem, we obtain

$$T^{1-H}\left(\frac{1}{T}\int_0^T X_t dt-\mu\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0,\frac{\sigma^2}{\kappa^2})\,. \tag{A.36}$$

Note that

$$T^{1-H}\left(\hat{\mu}_{HN}-\mu\right) = T^{1-H}\left(\hat{\mu}_{HN}-\frac{1}{T}\int_0^T X_t dt\right) + T^{1-H}\left(\frac{1}{T}\int_0^T X_t dt-\mu\right). \tag{A.37}$$

Using (2.9), (A.36) and (A.37), we obtain (3.6).

To obtain the asymptotic distribution of $\hat{\kappa}_{HN}$, from (1.1), we have

$$\int_0^T X_t dX_t = \kappa\mu\int_0^T X_t dt - \kappa\int_0^T X_t^2 dt + \sigma\int_0^T X_t dB_t^H\,.$$

Using the well-known results of the Stratonovich integral for fBm (**Weilin, I thought we have proposed to use the Ito-Skohorod integral here. Why change to the**

**Stratonovich integral?**) and the Malliavin derivative for $X_t$ (see, (14; 25; 18)), we can obtain

$$\kappa\mu\int_0^T X_t dt - \kappa\int_0^T X_t^2 dt + \sigma\int_0^T X_t dB_t^H = \frac{X_T^2 - X_0}{2} - \alpha_H\sigma^2\int_0^T\int_0^t u^{2H-2}e^{-\kappa u}dudt. \tag{A.38}$$

Combining (A.38) and the above equation, we deduce that

$$\int_0^T X_t dX_t = \frac{X_T^2 - X_0}{2} - \alpha_H\sigma^2\int_0^T\int_0^t u^{2H-2}e^{-\kappa u}dudt\,. \tag{A.39}$$

Based on (2.5) and (2.10), we can rewrite $\hat{\kappa}_{HN}$ as

$$\begin{aligned}\hat{\kappa}_{HN} &= \left(\frac{T^2\sigma^2 H\Gamma(2H)}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}\right)^{\frac{1}{2H}}\left(\frac{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}\right)^{\frac{1}{2H}} \quad (A.40)\\ &= \left(\frac{T^2\sigma^2 H\Gamma(2H)}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}\right)^{\frac{1}{2H}}\hat{\kappa}_{LS}^{\frac{1}{2H}}\,,\end{aligned}$$

Substituting (A.39) into (A.40), we have

$$\begin{aligned}\hat{\kappa}_{HN} &= \left(\frac{T^2\sigma^2 H\Gamma(2H)}{(X_T - X_0)\int_0^T X_t dt - T\left(\frac{1}{2}X_T^2 - \frac{1}{2}X_0^2 - \alpha_H\sigma^2\int_0^T\int_0^t u^{2H-2}e^{-\kappa u}dudt\right)}\right)^{\frac{1}{2H}}\hat{\kappa}_{LS}^{\frac{1}{2H}}\\ &= \left(\frac{\sigma^2 H\Gamma(2H)\hat{\kappa}_{LS}}{\frac{X_T}{T}\frac{1}{T}\int_0^T X_t dt - \frac{X_0}{T}\frac{1}{T}\int_0^T X_t dt - \frac{1}{2T}X_T^2 + \frac{1}{2T}X_0^2 + \alpha_H\sigma^2\frac{1}{T}\int_0^T\int_0^t u^{2H-2}e^{-\kappa u}dudt}\right)^{\frac{1}{2H}}.\end{aligned}$$

Hence,

$$\begin{aligned}&\sqrt{T}\left(\hat{\kappa}_{HN} - \kappa\right) \quad (A.41)\\ =\ &\sqrt{T}\left(\hat{\kappa}_{HN} - \kappa^{1-\frac{1}{2H}}\hat{\kappa}_{LS}^{\frac{1}{2H}} + \kappa^{1-\frac{1}{2H}}\hat{\kappa}_{LS}^{\frac{1}{2H}} - \kappa\right)\\ =\ &\sqrt{T}\left(\hat{\kappa}_{HN} - \kappa^{1-\frac{1}{2H}}\hat{\kappa}_{LS}^{\frac{1}{2H}}\right) + \sqrt{T}\kappa^{1-\frac{1}{2H}}\left(\hat{\kappa}_{LS}^{\frac{1}{2H}} - \kappa^{\frac{1}{2H}}\right)\\ =\ &\left[\left(\frac{\sigma^2 H\Gamma(2H)}{\frac{X_T}{T}\frac{1}{T}\int_0^T X_t dt - \frac{X_0}{T}\frac{1}{T}\int_0^T X_t dt - \frac{1}{2T}X_T^2 + \frac{1}{2T}X_0 + \alpha_H\sigma^2\frac{1}{T}\int_0^T\int_0^t u^{2H-2}e^{-\kappa u}dudt}\right)^{\frac{1}{2H}}\right.\\ &\left.-\kappa^{1-\frac{1}{2H}}\right]\sqrt{T}\hat{\kappa}_{LS}^{\frac{1}{2H}} + \sqrt{T}\kappa^{1-\frac{1}{2H}}\left(\hat{\kappa}_{LS}^{\frac{1}{2H}} - \kappa^{\frac{1}{2H}}\right)\,.\end{aligned}$$

By Theorem 3.2 and the delta method, we get

$$\sqrt{T}\left(\hat{\kappa}_{LS}^{\frac{1}{2H}} - \kappa^{\frac{1}{2H}}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \left(\frac{1}{2H}\kappa^{\frac{1-2H}{2H}}\right)^2\kappa C_H\right)\,. \tag{A.42}$$

By (A.9), Eq. (4.3) and Lemma 5.2 of (18), we can obtain

$$
\begin{aligned}
& \left(\frac{\sigma^2 H\Gamma\left(2H\right)}{\left(\frac{X_T}{T}-\frac{X_0}{T}\right)\frac{1}{T}\int_0^T X_t dt-\frac{1}{2T}X_T^2+\frac{1}{2T}X_0^2+\alpha_H\sigma^2\frac{1}{T}\int_0^T\int_0^t u^{2H-2}e^{-\kappa u}dudt}\right)^{\frac{1}{2H}} \qquad \text{(A.43)} \\
= \; & \kappa^{1-\frac{1}{2H}}+o_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}
$$

Finally, by Slutsky's theorem, Remark 3.2, (A.41), (A.42), and (A.43), we obtain the desired asymptotic distribution in (3.7).

## A.4. Proof of Lemma 3.1

Using (A.1), we obtain

$$
\begin{aligned}
& \frac{e^{\kappa T}}{T^H}\int_0^T X_t dB_t^H \qquad \text{(A.44)} \\
= \; & \frac{e^{\kappa T}}{T^H}\int_0^T\left[\left(1-e^{-\kappa t}\right)\mu+X_0e^{-\kappa t}+\sigma\int_0^t e^{-\kappa(t-s)}dB_s^H\right]dB_t^H \\
= \; & \frac{\mu e^{\kappa T}}{T^H}B_T^H+\frac{X_0-\mu}{T^H}e^{\kappa T}\int_0^T e^{-\kappa t}dB_t^H+\frac{\sigma e^{\kappa T}}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H .
\end{aligned}
$$

First, it is easy to see that

$$
\frac{\mu e^{\kappa T}}{T^H}B_T^H \xrightarrow{a.s.} 0 . \qquad \text{(A.45)}
$$

For $H\in(\frac{1}{2},1)$, from Lemma 6 of (3), we have

$$
\frac{X_0-\mu}{T^H}e^{\kappa T}\int_0^T e^{-\kappa t}dB_t^H \xrightarrow{a.s.} 0 . \qquad \text{(A.46)}
$$

Let us mention that (A.46) also follows obviously in the case $H=1/2$.

Next, we consider the last term of (A.44). If $H=1/2$, a simple calculation yields

$$
\begin{aligned}
\mathbb{E}\left[\frac{\sigma e^{\kappa T}}{T^{\frac{1}{4}}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s dB_t\right]^2 &= \frac{\sigma^2 e^{2\kappa T}}{T^{\frac{1}{2}}}\int_0^T\int_0^t e^{-2\kappa(t-s)}dsdt \qquad \text{(A.47)} \\
&= \frac{\sigma^2}{2\kappa}T^{\frac{1}{2}}e^{2\kappa T}+\frac{\sigma^2}{4\kappa^2\sqrt{T}}-\frac{\sigma^2 e^{2\kappa T}}{4\kappa^2\sqrt{T}} .
\end{aligned}
$$

If $H\in(\frac{1}{2},1)$, by the isometry property of the double stochastic integral, we have

$$
\mathbb{E}\left[\frac{\sigma e^{\kappa T}}{T^{\frac{H}{2}}}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H\right]^2=\sigma^2\alpha_H^2\frac{I_T}{e^{-2\kappa T}T^H} ,
$$

where

$$I_T = \int_{[0,T]^4} e^{-\kappa|v-s|}e^{-\kappa|u-r|}|u-v|^{2H-2}|r-s|^{2H-2}dudvdrds\,.$$

Taking the derivative of $I_T$ and $e^{-2\kappa T}T^H$ with respect to $T$, we have

$$\frac{dI_T}{d\left(e^{-2\kappa T}T^H\right)} = \frac{4\int_{[0,T]^3} e^{-\kappa(T-s)}e^{-\kappa|u-r|}(T-u)^{2H-2}|r-s|^{2H-2}dudrds}{HT^{H-1}e^{-2\kappa T}-2\kappa T^H e^{-2\kappa T}}\,.$$

By changing variables $T-s=x_1, T-r=x_2, T-u=x_3$, we get

$$\frac{dI_T}{d\left(e^{-2\kappa T}T^H\right)} = \frac{4\int_{[0,T]^3} e^{-\kappa x_1}e^{-\kappa|x_2-x_3|}x_3^{2H-2}|x_1-x_2|^{2H-2}dx_1dx_2dx_3}{HT^{H-1}e^{-2\kappa T}-2\kappa T^H e^{-2\kappa T}}\,.$$

Indeed, we can decompose the above integral into the integrals in the six disjoint regions $\{x_{\tau(1)}<x_{\tau(2)}<x_{\tau(3)}\}$, where $\tau$ runs over all permutations of the indices $\{1,2,3\}$. In the case $x_1<x_3<x_2$ making the change of variables $x_1=a$, $x_3-x_1=b$ and $x_2-x_3=c$ (other cases can be handled in a similar way), we obtain

$$\frac{dI_T}{d\left(e^{-2\kappa T}T^H\right)} = \frac{4\int_{[0,T]^3} e^{-\kappa a}e^{-\kappa c}\left(a+b\right)^{2H-2}\left(b+c\right)^{2H-2}dadbdc}{HT^{H-1}e^{-2\kappa T}-2\kappa T^H e^{-2\kappa T}}\,.$$

As a consequence,

$$\frac{dI_T}{d\left(e^{-2\kappa T}T^H\right)} \le \frac{4\int_{[0,T]^3} e^{-\kappa(a+c)}b^{4H-4}dadbdb}{HT^{H-1}e^{-2\kappa T}-2\kappa T^H e^{-2\kappa T}}\,. \tag{A.48}$$

Then, from (A.47) - (A.48), we obtain

$$|\frac{\sigma e^{\kappa T}}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H|_{L^2(\Omega)} \le CT^{-\frac{H}{2}}, \tag{A.49}$$

with $H\in[\frac{1}{2},1)$ and $C$ denotes a suitable positive constant.

Consequently, we deduce from (A.49) and Lemma 2.1 of (21) that

$$\frac{\sigma e^{\kappa T}}{T^H}\int_0^T\int_0^t e^{-\kappa(t-s)}dB_s^H dB_t^H \overset{a.s.}{\to} 0\,.$$

Finally, the result in Lemma 3.1 follows by combining (A.44), (A.45), (A.46) and (4).

## A.5. Proof of Theorem 3.4

We prove the convergence of $\hat{\kappa}_{LS}$ first. For the sake of simple notations, we introduce the two processes with $T\ge 0$

$$Z_T = \int_0^T e^{\kappa s}B_s^H ds\,, \tag{A.50}$$

$$\xi_T = \int_0^T e^{\kappa s} dB_s^H \,. \tag{A.51}$$

By the definition of the Young integral, $B_0^H = 0$, and the definition of $Z_T$, we can write

$$\xi_T = e^{\kappa T} B_T^H - \kappa \int_0^T e^{\kappa s} B_s^H ds = e^{\kappa T} B_T^H - \kappa Z_T \,. \tag{A.52}$$

By Lemma 2.1 of (15), we obtain $Z_\infty = \int_0^\infty e^{\kappa s} B_s^H ds$ which is well-defined and

$$\begin{aligned} Z_T &\overset{a.s.}{\to} Z_\infty \,, && \text{(A.53)} \\ \xi_T &\overset{a.s.}{\to} \xi_\infty := -\kappa Z_\infty \,. && \text{(A.54)} \end{aligned}$$

Using (A.50) and the Young integral, we can rewrite the solution of (1.1) as

$$\begin{aligned} X_t &= X_0 e^{-\kappa t} + (1 - e^{-\kappa t})\mu + e^{-\kappa t} \sigma \int_0^t e^{\kappa s} dB_s^H && \text{(A.55)} \\ &= X_0 e^{-\kappa t} + (1 - e^{-\kappa t})\mu + e^{-\kappa t} \sigma \xi_t \\ &= X_0 e^{-\kappa t} + (1 - e^{-\kappa t})\mu + e^{-\kappa t} \sigma \left[ e^{\kappa t} B_t^H - \int_0^t B_s^H e^{\kappa s} \kappa ds \right] \\ &= X_0 e^{-\kappa t} + (1 - e^{-\kappa t})\mu + \sigma B_t^H - \sigma e^{-\kappa t} \kappa \int_0^t B_s^H e^{\kappa s} ds \\ &= X_0 e^{-\kappa t} + (1 - e^{-\kappa t})\mu + \sigma B_t^H - \sigma e^{-\kappa t} \kappa Z_t \,. \end{aligned}$$

To prove the consistency of $\hat{\kappa}_{LS}$, we will analyze separately the numerator and the denominator of the estimator (3.8). First, we consider the term $e^{\kappa T} \int_0^T X_t dt$. Using L'Hôspital's rule, (A.53), (A.54) and (A.55), we obtain

$$\begin{aligned} e^{\kappa T} \int_0^T X_t dt &= e^{\kappa T} \int_0^T \left[ X_0 e^{-\kappa t} + \left(1 - e^{-\kappa t}\right) \mu + \sigma e^{-\kappa t} \xi_t \right] dt && \text{(A.56)} \\ &= -\frac{X_0}{\kappa} \left(1 - e^{\kappa T}\right) + e^{\kappa T} \mu T + \frac{\mu}{\kappa} e^{\kappa T} \left(e^{-\kappa T} - 1\right) + \sigma \frac{\int_0^T e^{-\kappa t} \xi_t dt}{e^{-\kappa T}} \\ &\overset{a.s.}{\to} -\frac{X_0}{\kappa} + \frac{\mu}{\kappa} + \sigma Z_\infty \,. \end{aligned}$$

Combining (A.53), (A.54) and (A.55), we deduce that

$$\begin{aligned} \frac{1}{T} e^{\kappa T} X_T &= \frac{e^{\kappa T}}{T} \left[ X_0 e^{-\kappa T} + \left(1 - e^{-\kappa T}\right) \mu + \sigma e^{-\kappa T} \xi_T \right] && \text{(A.57)} \\ &= \frac{1}{T} \left[ X_0 + \mu e^{\kappa T} - \mu + \sigma \xi_T \right] \\ &\overset{a.s.}{\to} 0. \end{aligned}$$

By (A.53) and (A.54), we have

$$\begin{aligned} X_T^2 e^{2\kappa T} &= e^{2\kappa T}\left[X_0 e^{-\kappa T} + \left(1-e^{-\kappa T}\right)\mu + \sigma e^{-\kappa T}\xi_T\right]^2 \qquad (A.58)\\ &= e^{2\kappa T}\Big[\left(X_0 e^{-\kappa T}\right)^2 + \left(1-e^{-\kappa T}\right)^2\mu^2 + \sigma^2 e^{-2\kappa T}\xi_T^2 + 2X_0 e^{-\kappa T}\sigma e^{-\kappa T}\xi_T \\ &\quad +2\mu\left(1-e^{-\kappa T}\right)\sigma e^{-\kappa T}\xi_T + 2X_0 e^{-\kappa T}\left(1-e^{-\kappa T}\right)\mu\Big] \\ &= X_0^2 + \left(e^{\kappa T}-1\right)^2\mu^2 + \sigma^2\xi_T^2 + 2X_0\sigma\xi_T + 2\mu\sigma\xi_T(e^{\kappa T}-1) + 2\mu X_0\left(e^{\kappa T}-1\right) \\ &\overset{a.s.}{\to} X_0^2 + \mu^2 + \sigma^2\kappa^2 Z_\infty^2 - 2\sigma X_0\kappa Z_\infty + 2\mu\sigma\kappa Z_\infty - 2X_0\mu\,. \end{aligned}$$

By (A.53) and (A.54) again, we obtain

$$\begin{aligned} e^{2\kappa T}\int_0^T X_t^2 dt &= e^{2\kappa T}\int_0^T \left[X_0 e^{-\kappa t} + \left(1-e^{-\kappa t}\right)\mu + \sigma e^{-\kappa t}\xi_t\right]^2 dt \\ &= e^{2\kappa T}X_0\int_0^T e^{-2\kappa t}dt + e^{2\kappa T}\int_0^T \mu^2(1-e^{-\kappa t})^2 dt + \sigma^2 e^{2\kappa T}\int_0^T e^{-2\kappa t}\xi_t^2 dt \\ &\quad +2e^{2\kappa T}\mu X_0\int_0^T e^{-\kappa t}\left(1-e^{-\kappa t}\right)dt + 2e^{2\kappa T}X_0\sigma\int_0^T e^{-2\kappa t}\xi_t dt \\ &\quad +2e^{2\kappa T}\mu\sigma\int_0^T \left(1-e^{-\kappa t}\right)e^{-\kappa t}\xi_t dt \\ &= \frac{X_0}{2\kappa}\left(e^{2\kappa T}-1\right) + \mu^2\left[e^{2\kappa T}T - \frac{1}{2\kappa}(1-e^{2\kappa T}) + \frac{2}{\kappa}(e^{\kappa T}-e^{2\kappa T})\right] \\ &\quad +\sigma^2 e^{2\kappa T}\int_0^T e^{-2\kappa t}\xi_t^2 dt + 2\mu X_0\left[\frac{1}{2\kappa}(1-e^{2\kappa T}) - \frac{1}{\kappa}\left(e^{\kappa T}-e^{2\kappa T}\right)\right] \\ &\quad +2\sigma X_0\frac{\int_0^T e^{-2\kappa t}\xi_t dt}{e^{-2\kappa T}} + 2\mu\sigma\left(\frac{\int_0^T e^{-\kappa t}\xi_t dt}{e^{-2\kappa T}} - \frac{\int_0^T e^{-2\kappa t}\xi_t dt}{e^{-2\kappa T}}\right) \\ &\overset{a.s.}{\to} -\frac{X_0}{2\kappa} - \frac{\mu^2}{2\kappa} - \frac{\sigma^2}{2}\kappa Z_\infty^2 + \frac{\mu X_0}{\kappa} + X_0\sigma Z_\infty - \mu\sigma Z_\infty\,. \qquad (A.59) \end{aligned}$$

A standard calculation together with (A.56) yields

$$\frac{e^{2\kappa T}}{T}\left(\int_0^T X_t dt\right)^2 = \frac{1}{T}\left(e^{\kappa T}\int_0^T X_t dt\right)^2 \overset{a.s.}{\to} 0\,. \qquad (A.60)$$

Combining (A.56), (A.57), (A.58), (A.59), (A.60) and (3.8), we obtain the strong consistency of $\hat{\kappa}_{LS}$.

It remains to show the almost sure convergence of $\hat{\mu}_{LS}$. From (1.1) and the fact that $B_0^H = 0$, we can rewrite $X_t$ as

$$X_t = X_0 + \mu\kappa t - \kappa\int_0^t X_s ds + \sigma B_t^H\,. \qquad (A.61)$$

By (1.1), (3.9), (A.61) and the Young integral, we can rewrite $\hat{\mu}_{LS}$ as

$$\begin{aligned}
\hat{\mu}_{LS} &= \frac{(X_T - X_0)\int_0^T X_t^2 dt - \int_0^T X_t dX_t \frac{X_0 + \mu\kappa T + \sigma B_T^H - X_T}{\kappa}}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t} \qquad \text{(A.62)}\\
&= \frac{(X_T - X_0)\int_0^T X_t^2 dt - \mu T\int_0^T X_t dX_t - \frac{X_0 + \sigma B_T^H - X_T}{\kappa}\int_0^T X_t\left[\kappa(\mu - X_t)\,dt + \sigma dB_t^H\right]}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}\\
&= \mu + \frac{\frac{X_T - X_0}{\kappa}\sigma\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa}\int_0^T X_t dX_t}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t}\\
&= \mu + \frac{e^{\kappa T}\frac{X_T - X_0}{\kappa}\frac{\sigma}{T}e^{\kappa T}\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa T}e^{2\kappa T}\frac{X_T^2 - X_0^2}{2}}{e^{\kappa T}\frac{X_T - X_0}{T}e^{\kappa T}\int_0^T X_t dt - e^{2\kappa T}\frac{X_T^2 - X_0^2}{2}}.
\end{aligned}$$

Finally, using (A.56), (A.57), (A.58), Lemma 3.1 and (A.62), we obtain the desired strong consistency for $\hat{\mu}_{HN}$. (**Weilin, this is not entirely clear to me.**)

## A.6. Proof of Theorem 3.5

Using (1.1), (A.61) and the Young integral, we can rewrite $\hat{\kappa}_{LS}$ as

$$\begin{aligned}
\hat{\kappa}_{LS} &= \frac{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t\left[\kappa(\mu - X_t)\,dt + \sigma dB_t^H\right]}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \qquad \text{(A.63)}\\
&= \frac{(X_T - X_0 - \kappa\mu T)\int_0^T X_t dt + \kappa T\int_0^T X_t^2 dt - \sigma T\int_0^T X_t dB_t^H}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}\\
&= \kappa + \frac{\sigma B_T^H\int_0^T X_t dt - \sigma T\int_0^T X_t dB_t^H}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
e^{-\kappa T}(\hat{\kappa}_{LS} - \kappa) &= \frac{\sigma B_T^H e^{-\kappa T}\int_0^T X_t dt - \sigma T e^{-\kappa T}\int_0^T X_t dB_t^H}{T\int_0^T X_t^2 dt - \left(\int_0^T X_t dt\right)^2} \qquad \text{(A.64)}\\
&= \frac{\frac{\sigma B_T^H}{T}\frac{e^{\kappa T}\int_0^T X_t dt}{e^{2\kappa T}\int_0^T X_t^2 dt} - \frac{\sigma e^{\kappa T}\int_0^T X_t dB_t}{e^{2\kappa T}\int_0^T X_t^2 dt}}{1 - \frac{1}{T}\frac{\left(e^{\kappa T}\int_0^T X_t dt\right)^2}{e^{2\kappa T}\int_0^T X_t^2 dt}}.
\end{aligned}$$

A standard calculation yields

$$
\begin{aligned}
-\frac{\sigma e^{\kappa T}\int_0^T X_t dB_t^H}{e^{2\kappa T}\int_0^T X_t^2 dt} &= -\sigma\frac{e^{\kappa T}\int_0^T\left[\mu+(X_0-\mu)\,e^{-\kappa t}+\sigma\int_0^t e^{-\kappa(t-s)}dB_s^H\right]dB_t^H}{e^{2\kappa T}\int_0^T X_t^2 dt} \quad \text{(A.65)}\\
&= -\frac{\sigma}{e^{2\kappa T}\int_0^T X_t^2 dt}\Bigg[\mu e^{\kappa T}B_T^H-\sigma e^{\kappa T}\int_0^T\int_0^s e^{-\kappa(t-s)}dB_t^H dB_s^H\\
&\quad +e^{\kappa T}\int_0^T e^{-\kappa t}dB_t^H\left[(X_0-\mu)+\sigma\int_0^T e^{\kappa s}dB_s^H\right]\Bigg].
\end{aligned}
$$

By Lemma 6 and Lemma 3 of (3), we have

$$
e^{\kappa T}\int_0^T e^{-\kappa s}dB_s^H \xrightarrow{\mathcal{L}} \mathcal{N}\left(0,\frac{H\Gamma(2H)}{|\kappa|^{2H}}\right), \quad \text{(A.66)}
$$

$$
\int_0^T e^{\kappa s}dB_s^H \xrightarrow{\mathcal{L}} \mathcal{N}\left(0,\frac{H\Gamma(2H)}{|\kappa|^{2H}}\right). \quad \text{(A.67)}
$$

Moreover, it is easy to check

$$
\mu e^{\kappa T}B_T^H \xrightarrow{a.s.} 0\,. \quad \text{(A.68)}
$$

Obviously, both $e^{\kappa t}$ and $e^{\kappa s}$ are non-random Hölder continuous functions. According to Lemma 7 of (3) and the relationship between the divergence integral and path-wise integral (see e.g. Eq. (2.4) in (3)), we can deduce that

$$
\sigma e^{\kappa T}\int_0^T\int_0^t e^{-\kappa s}dB_s^H e^{\kappa t}dB_t^H \xrightarrow{p} 0\,. \quad \text{(A.69)}
$$

By (A.59), (A.65) - (A.69) and Slutsky's theorem, we have

$$
-\frac{\sigma e^{\kappa T}\int_0^T X_t dB_t^H}{e^{2\kappa T}\int_0^T X_t^2 dt} \xrightarrow{\mathcal{L}} \frac{2\kappa\sigma\frac{\sqrt{H\Gamma(2H)}}{|\kappa|^H}\nu}{X_0-\mu+\sigma\frac{\sqrt{H\Gamma(2H)}}{|\kappa|^H}\omega}, \quad \text{(A.70)}
$$

with $\nu$ and $\omega$ being two independent standard normal random variables. Combining (A.56), (A.59), (A.64), and (A.70), we obtain (3.10).

Let us now obtain the asymptotic distribution of $\hat{\mu}_{LS}$. From (A.62), we can rewrite

$\hat{\mu}_{LS}$ as

$$\begin{aligned}
\hat{\mu}_{LS} &= \frac{(X_T - X_0)\int_0^T X_t^2 dt - \int_0^T X_t dX_t \frac{X_0 + \mu\kappa T + \sigma B_T^H - X_T}{\kappa}}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t} \\
&= \frac{(X_T - X_0)\int_0^T X_t^2 dt - \mu T\int_0^T X_t dX_t - \frac{X_0 + \sigma B_T^H - X_T}{\kappa}\int_0^T X_t\left[\kappa\left(\mu - X_t\right)dt + \sigma dB_t^H\right]}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t} \\
&= \mu + \frac{\frac{X_T - X_0}{\kappa}\sigma\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa}\int_0^T X_t dX_t}{(X_T - X_0)\int_0^T X_t dt - T\int_0^T X_t dX_t} \\
&= \mu + \frac{\frac{X_T - X_0}{\kappa}\sigma\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa}\frac{X_T^2 - X_0^2}{2}}{(X_T - X_0)\int_0^T X_t dt - T\frac{X_T^2 - X_0^2}{2}}.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
T^{1-H}\left(\hat{\mu}_{LS} - \mu\right) &= \frac{\frac{2\sigma}{\kappa T^H X_T}\int_0^T X_t dB_t^H - \frac{2X_0\sigma}{\kappa T^H X_T^2}\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa T^H} + \frac{\sigma B_T^H}{\kappa T^H}\frac{X_0^2}{X_T^2}}{\frac{2}{TX_T}\int_0^T X_t dt - \frac{2X_0}{TX_T^2}\int_0^T X_t dt - 1 + \frac{X_0}{X_T^2}} \\
&= \frac{\frac{2\sigma}{\kappa e^{\kappa T} X_T}\frac{e^{\kappa T}}{T^H}\int_0^T X_t dB_t^H - \frac{2X_0\sigma}{\kappa X_T^2 e^{2\kappa T}}\frac{e^{2\kappa T}}{T^H}\int_0^T X_t dB_t^H - \frac{\sigma B_T^H}{\kappa T^H} + \frac{\sigma}{\kappa}\frac{e^{2\kappa T}X_0^2}{e^{2\kappa T}X_T^2}\frac{B_T^H}{T^H}}{\frac{2}{Te^{\kappa T}X_T}e^{\kappa T}\int_0^T X_t dt - \frac{2X_0}{Te^{2\kappa T}X_T^2}e^{2\kappa T}\int_0^T X_t dt - 1 + \frac{e^{2\kappa T}X_0^2}{e^{2\kappa T}X_T^2}}.
\end{aligned}$$

By (A.56) - (A.58), (A.34), Lemma 3.1, and the above equation, we can obtain the desired result in (3.11).

## A.7. Proof of Theorem 3.6

By the law of the iterated logarithm for fBm (see e.g. (34)), we get

$$\frac{B_T^H \int_0^T B_t^H dt - T\int_0^T B_t^H dB_t^H}{T\int_0^T (B_t^H)^2 dt - \left(\int_0^T B_t^H dt\right)^2} \xrightarrow{a.s.} 0. \tag{A.71}$$

As a consequence, we obtain the almost sure convergence from (3.12) and (A.71). By the scaling properties of fBm of (2.2) (see also in (25)), we have

$$\begin{cases}
B_T^H \stackrel{d}{=} T^H B_1^H\,, \\
B_T^H \int_0^T B_t^H dt \stackrel{d}{=} T^{2H+1} B_1^H \int_0^1 B_u^H du\,, \\
T\int_0^T B_t^H dB_t^H \stackrel{d}{=} T^{2H+1}\int_0^1 B_u^H dB_u^H\,, \\
T\int_0^T (B_t^H)^2 dt \stackrel{d}{=} T^{2H+2}\int_0^1 (B_u^H)^2 du\,, \\
\left(\int_0^T B_t^H dt\right)^2 \stackrel{d}{=} T^{2H+2}\left(\int_0^1 B_u^H du\right)^2\,,
\end{cases} ; \tag{A.72}$$

Combining (3.12) and (A.72), we can obtain the desired asymptotic distribution.